Exploring the Penrose Tiling

Qasim Salahuddin

Introduction

A tiling of the plane is said to be non-periodic if there exists no translation which maps the tiling back onto itself. A set of tiles is said to be aperiodic if they can tile the place, but only non-periodically.

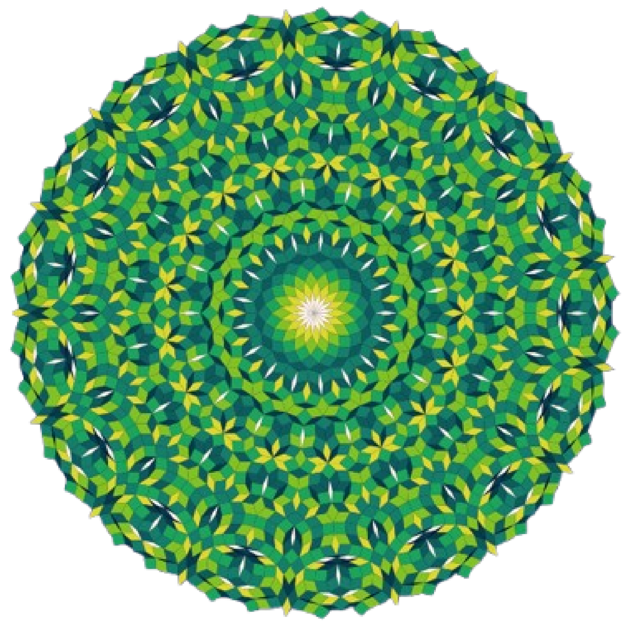


Figure 1: A non-periodic tiling related to the Penrose tiling, made from an aperiodic set of size 17. [1]

The first aperiodic set to be discovered, in 1966, were a set of 20,426 'Wang' tiles. This set was later reduced to 11. In 1971, Raphael Robinson found an aperiodic set consisting of 6 tiles. Then in 1974 Roger Penrose discovered the famous Penrose tiling, consisting of only 2 different tiles. Finally, in 2023, a single tile was found, known as an 'Einstein' tile, which by itself only tiles the plane non-periodically [2]. In this poster we will explore in depth the Penrose tiles, specifically the Penrose 'Kite and Dart'.

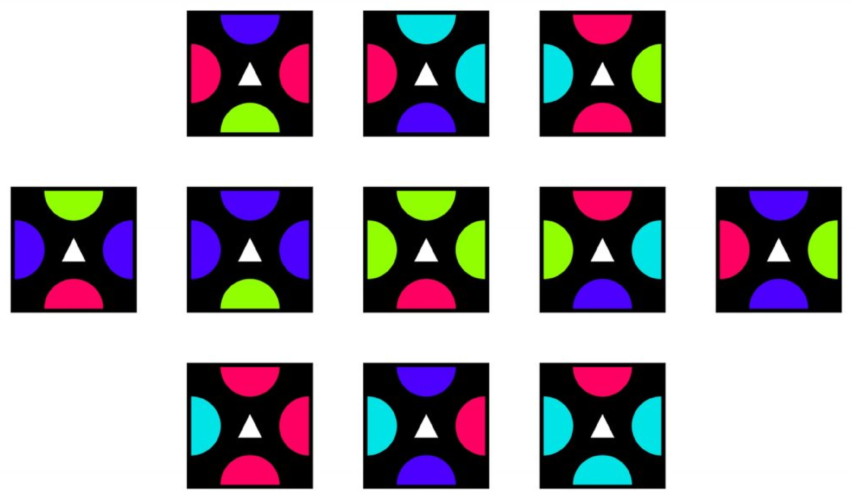


Figure 2: An aperiodic set of 11 Wang tiles [3].

Preliminaries

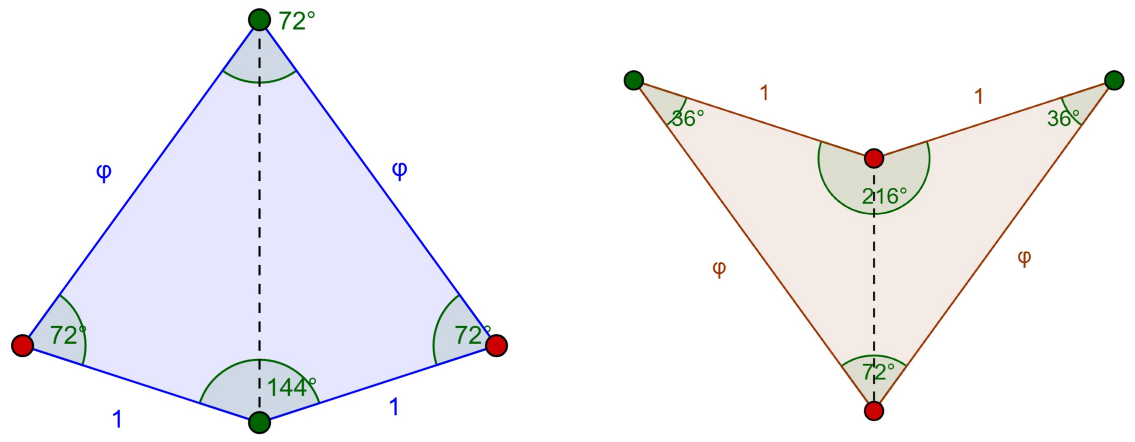


Figure 3: The Penrose Kite and Dart. ϕ denotes the golden ratio. [4]

Above we can see the Penrose 'Kite and Dart' tiles. Note that the vertices have each been assigned a colour, and we require that all touching vertices share the same colour. Each shape can be split along the dotted lines into triangles. We investigate these triangles further:

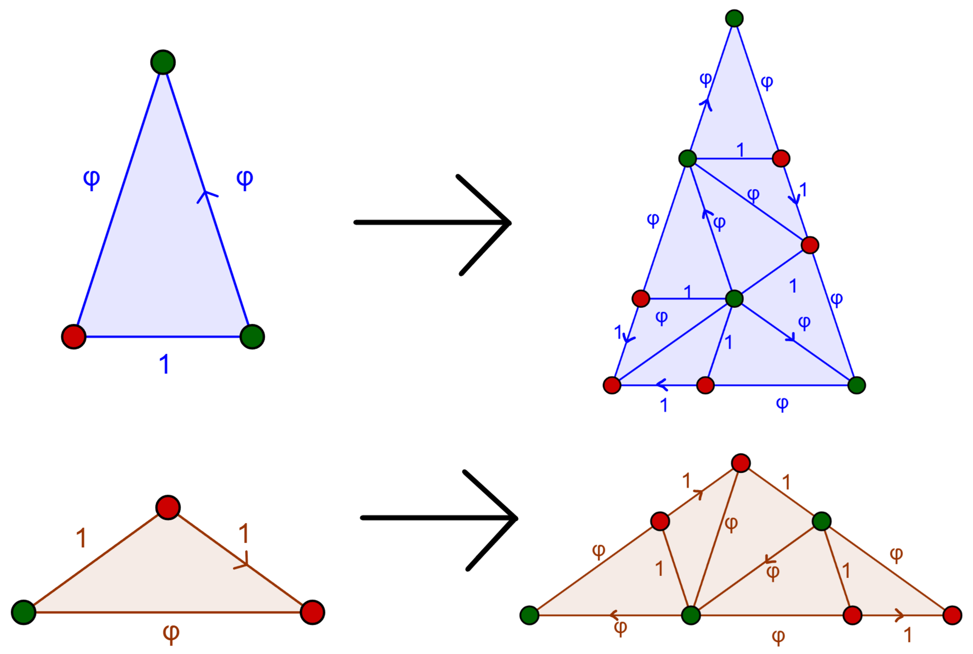


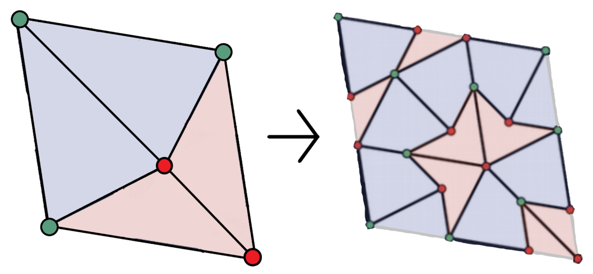
Figure 4: The triangles can be decomposed into congruent triangles.

The 2 triangles can be 'inflated' into similar triangles, each made out of triangles congruent to the originals [5]. The scale factor between the enlargement is ϕ^2. The matching conditions on these enlargements remain the same. To ensure that the triangles tile back into the kite and dart, we add the conditions that edges can only match up with other edges of the same size, and with arrows pointing in the same direction. Note that this make up is **unique**.

Extension Theorem: If a (finite) set of tiles can tile an arbitrarily large disk, they can tile the entire plane [5]. (Proof of this theorem is beyond the scope of this poster)

Proof: The 'Kite and Dart' tile the plane

Start with any single triangle. We can now decompose and inflate this triangle using the process shown in the previous section. We can then decompose all the resulting triangles, creating an even larger tiling. This process can continue to be repeated, tiling arbitrarily large areas, so by the Extension Theorem the triangles tile the plane. The matching conditions force us to make kite and darts in any tiling, hence the Penrose tiles tile the plane [5].



Proof: The 'Kite and Dart' are aperiodic

Assume that there exists a translation, say of length D, which maps the tiling back onto itself. Denote the radius of the largest circle to fit inside any tile as r. Then if we inflate the tiling, this translation must also map this new tiling back onto itself (by uniqueness of the decomposition). The radius of the largest circle to fit in any tile is now $r\phi^2$. We repeat n times, such that $r\phi^{2n} > D$. But now the translation cannot map back onto itself, as there will be a tile which will not leave itself, so no such translation exists. Hence the tiling is non-periodic, and the triangles form an aperiodic set. Since the matching conditions force us to form kite and darts with the triangles, it follows that the Penrose tiles are aperiodic [5].

Applications

- Quasicrystals are chemical compounds akin to the 3D equivalent of an aperiodic tiling. Specifically, they are comparable to the Penrose tiling in the sense that they both have 5-fold symmetries, a phenomena deemed to be impossible until very recently [6].
- In 1997 Kleenex had the idea of imprinting the Penrose tiling onto their toilet rolls. The idea was that due to the non-periodic property of the tiling, there would be no point where the paper would line up perfectly with one another when rolled up, hence making the final roll thicker. Unfortunately, Penrose had a patent on the tiling, and Kleenex were forced to remove the design after a court battle [5].

Figure 5: Left: A roll of toilet paper featuring the Penrose tiling [7]. Right: A quasicrystalline structure [8].

References

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[8] By LinKayser - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=147557643